EQUATION OF MOTION

In principle, the relationships explaining motion in the ocean can be written in a series of mathematical equations. In practice, we can only evaluate these equations in part; thus those equations most commonly used in representing currents, waves, tides, turbulence, and other forms of motion are at best fair approximations of what occurs. In this chapter we consider the equations of motion and how they apply to understanding some aspects of ocean circulation.

Any quantitative discussion of forces and motions requires a coordinate system. The system most commonly used in oceanography is the rectilinear Cartesian system in which the earth is assumed to be flat. A spherical coordinate system would be more realistic, but it is also more complicated. The Cartesian system has been found adequate for nearly all problems in physical oceanography.

The usual convention is to assume a plane in which the x axis points east, the y axis points north, and the z axis is up; more precisely, the z axis is in the direction opposite to the gravitational vector. The corresponding velocity vectors are u, v, and w.

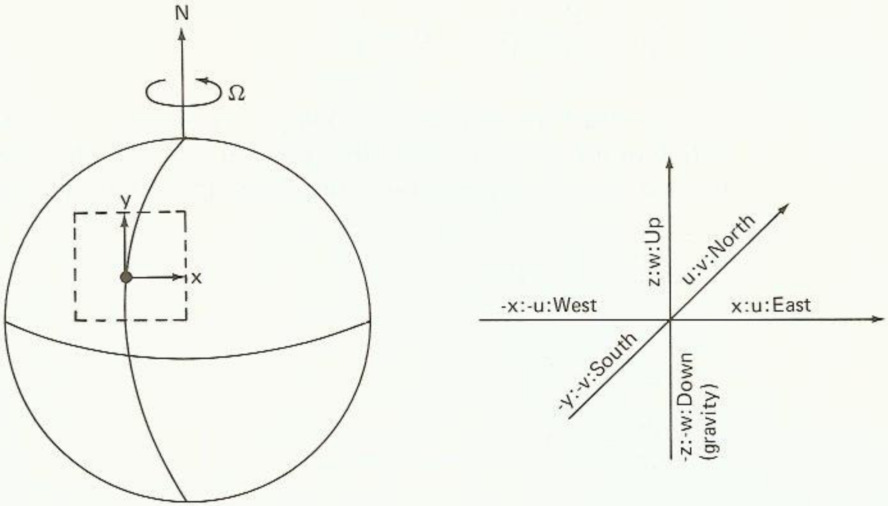


Figure 5.1. Cartesian coordinate system used in this text.

Although meteorologists and oceanographers may agree on the coordinate system in Figure 5.1, they use different conventions for describing winds and currents. A north current is current flowing toward the north; a north wind is a wind blowing from the north. The convention is confusing, but there is little likelihood that it will be changed. To minimize the confusion, this text refers to northerly winds and northward currents.

Newton's second law states that the acceleration of a particle is proportional to the sum of the forces acting on the particle.

$$\frac{du}{dt} = \frac{1}{m} \Sigma \mathrm{F} \qquad (5.1)$$

In discussing fluid motion, the relationship is usually written

$$\frac{du}{dt} = \frac{1}{\rho} \Sigma \mathrm{F} \qquad (5.2)$$

where it is now understood that the forces are "per unit volume," since Eq. (5.2) can be derived from Eq. (5.1):

$$\frac{du}{dt} = \frac{V}{m} \Sigma \frac{F}{V}, \qquad \rho = \frac{m}{V} \tag{5.3}$$

As written, Eqs. (5.1) and (5.2) apply to the components of the forces acting in the east-west or x direction. Similar equations can be written for the force components acting along the other two axes:

$$\begin{aligned} \frac{du}{dt} &= \frac{1}{\rho} \Sigma F_x \\ \frac{dv}{dt} &= \frac{1}{\rho} \Sigma F_y \\ \frac{dw}{dt} &= \frac{1}{\rho} \Sigma F_z \end{aligned} \tag{5.4}$$

There are four important kinds of forces acting on a fluid particle in the ocean: gravitational, pressure gradient, frictional, and Coriolis. In a generalized way, Eq. (5.2) may be written

$$\text{density} \times \text{particle acceleration} = \text{gravity} + \text{pressure gradient} + \text{Coriolis} + \text{friction} \tag{5.5}$$

The mathematical expression for the forces of gravity, pressure, and Coriolis may be expressed quite simply. The various forms of the frictional forces are less easy to express in a precise manner, and they are considerably more difficult to measure in the ocean. The two problems are not unrelated. Note that, by choosing a coordinate system such that the z axis points opposite to the direction of gravity, there will be no gravitational force in either the x or y direction.

Pressure Gradient

Of the various terms in Eq. (5.5), perhaps the pressure gradient is the easiest to visualize. A particle will move from high pressure to low pressure, and the acceleration is simply proportional to the pressure gradient. A mechanical analog is a ball on a frictionless inclined plane. The ball rolls down the plane (from high to low pressure), and the acceleration of the ball is proportional to the inclination of the plane (pressure gradient). Mathematically, Eq. (5.5) now becomes (see Appendix I)

$$\begin{aligned} \frac{du}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \text{other forces} \\ \frac{dv}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \text{other forces} \\ \frac{dw}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \text{other forces} \end{aligned} \tag{5.8}$$

Pressure gradients arise in a variety of ways. One of the simplest is by

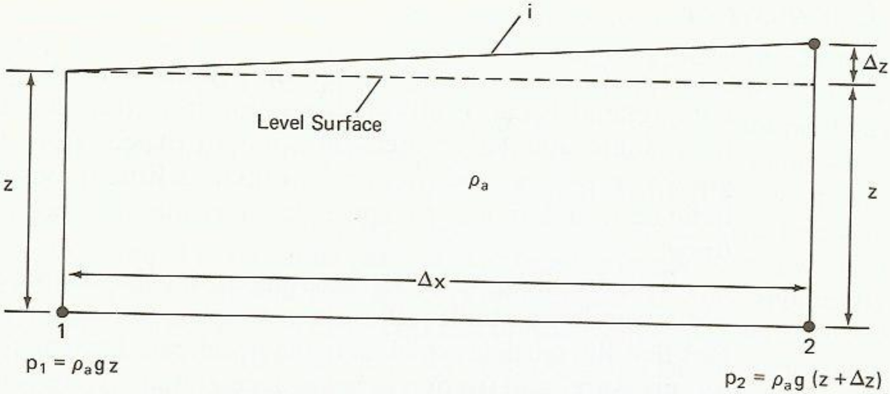


Figure 5.3. Slope of the sea surface creates a pressure gradient, Eq. (5.10).

a sloping water surface. Imagine a container with an ideal fluid whose density is ρ_a, and that in some manner it is possible to have the water surface slope as in Figure 5.3 without causing any other motion. Remembering that the pressure at any point in the fluid is simply the weight,

$$\begin{aligned} p_1 &= \rho_a g Z \\ p_2 &= \rho_a g(Z + \Delta Z) \end{aligned} \tag{5.9}$$

The resulting pressure-gradient term is

$$\begin{aligned} \frac{1}{\rho}\frac{\partial p}{\partial x} &= \frac{1}{\rho_a}\frac{p_2 - p_1}{\Delta X} \\ &= g\,\frac{\Delta Z}{\Delta X} \\ &= gi \end{aligned} \tag{5.10}$$

where i is the slope of the fluid surface.

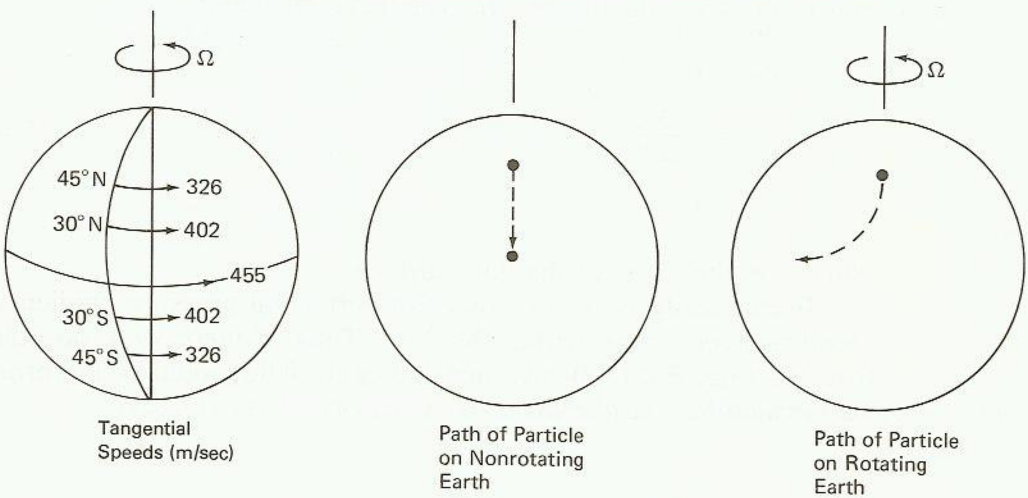
It can easily be shown that the horizontal pressure gradient is identical everywhere within the fluid. Thus, if there were no other forces acting, Eq. (5.8) says that the entire fluid would be uniformly accelerated toward the lower pressure.

Coriolis Force

The Coriolis "force" is the most difficult of the four forces to comprehend because physical intuition is of little avail. Most of us have some qualitative ideas of what to expect from the forces of gravity, pressure, and friction; but there is little in our experience to indicate what happens to a particle under the influence of the Coriolis force.

The first thing to understand about the Coriolis force is that it is not a true force at all; rather, it is a device for compensating for the fact that the particle which is being accelerated by the forces of gravity, pressure, and friction is being accelerated on a rotating earth, and all measurements of forces and accelerations are made relative to a rotating earth. Two examples will indicate the nature of the problem. The earth, with a radius of about 6400 km which rotates once every 24 h, has tangential velocities as indicated in Figure 5.4. Let us assume that a particle of water is set in motion at 45°N with a southward velocity of 1m/s, and that no forces other than gravity act upon it. According to Newton's first law, a particle in motion will continue to move at a constant velocity in the absence of any force. Thus, in slightly less than 2 days the particle should pass 30°N, continuing its southward journey at 1 m/s. However, the 1-m/s velocity is measured

Figure 5.4. Because of the change of tangential speed with latitude, a particle moving toward the equator appears to be accelerated to the west.



relative to the earth. In terms of a coordinate system that allows for earth rotation, the particle also has an eastward velocity of 326 m/s. However, the eastward tangential velocity at 30°N is 402 m/s. Thus to an observer on the earth it would appear that the particle not only has the 1-m/s southward component, but it would also be traveling westward at a speed of 76 m/s. To an observer on the earth it would appear that the particle had undergone a tremendous westward acceleration.

One can play the same game by starting the particle northward at 30°N at 1 m/s and find that at 45°N the particle is now apparently traveling eastward at 76 m/s. You can do the same thing between 30° and 45°S and find that the east-west velocities are of the same magnitude, but that the apparent acceleration is in the opposite sense. As you move your imaginary particle north and south along the earth you can find a general rule: in the Northern Hemisphere the apparent acceleration is always to the right of the direction in which the particle is moving; in the Southern Hemisphere the apparent acceleration is to the left of the direction of flow. At the equator, the acceleration will go through an inflection point; there will be no acceleration at the equator.

Hydrostatic Equation

The hydrostatic equation relates the static pressure within a fluid to the overlying weight of this fluid. The pressure at some depth $-Z$ is simply

$$p = -\int_0^{-Z} \rho g \, dz \tag{6.1}$$

and in the case where density is constant

$$p = +\rho g Z \tag{6.2}$$

The hydrostatic equation can be considered a special case of the vertical component of the equation of motion, Eq. (5.26), where there is no friction or vertical acceleration. In this case Eq. (5.26) reduces to

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g \tag{6.3}$$

which can be integrated to give Eq. (6.1).

The hydrostatic approximation is a quite general solution to the vertical component of Eq. (5.26). One case where it is not is when one considers wave motion. In that case the vertical component of acceleration must be included (see Chapter 10 and Appendix I).

Since the density of seawater is slightly greater than unity, and the acceleration of gravity is slightly less than 1000 cm/s^2, the pressure of 10 m of seawater is very close to 1 bar (1 million dynes/cm^2), which is the approximate pressure of the standard atmosphere. Thus every increase in depth of 10 m is the equivalent of increasing the pressure about 1 atmosphere.

Because of this simple numerical relationship, oceanic pressure is often measured in decibars (one tenth of a bar), since 1 decibar is approximately equivalent to 1 m in depth. Most "depth measurements" within the water column are measured by pressure-recording devices of one kind or another. Usually, these are calibrated in terms of 1 m equals 1 decibar. Although the agreement deteriorates slightly at great depths as compressibility increases the density of seawater, the differences are relatively small. The pressure gauge on the bathyscaphe *Trieste* on her dive to the bottom of the Marianas Trench recorded a pressure of 11,240 decibars. Systematic soundings in the area indicated a maximum depth of about 10,880 m for the trench.

The hydrostatic equation can be used in simplifying the pressure-gradient term in Eq. (5.26). Consider first a fluid of constant density. If the surface of the fluid were level, there would be no horizontal pressure gradient; but if the fluid surface sloped, there would be a horizontal pressure gradient.

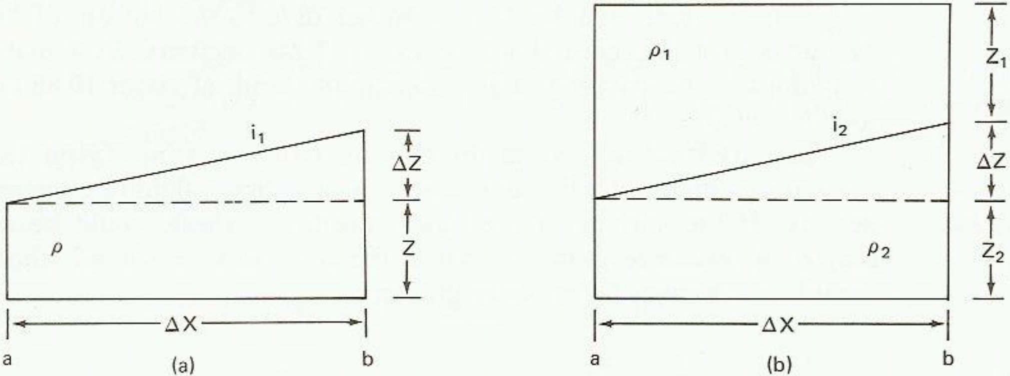
$$\begin{aligned}\frac{1}{\rho}\frac{\partial p}{\partial x} &= \frac{1}{\rho}\frac{p_b - p_a}{\Delta X}\\ &= \frac{1}{\rho \Delta X}[\rho g(Z + \Delta Z) - \rho g Z]\\ &= g\frac{\Delta Z}{\Delta X}\\ &= g i_1\end{aligned} \tag{6.4}$$

where i_1 is the slope of the surface in the x direction (Figure 6.1a). It is easy to demonstrate that this pressure gradient applies through the entire fluid.

Consider next a two-layer fluid. The upper surface is level, but there is a slope to the interface (Figure 6.1b). Then the pressure-gradient term in the bottom layer becomes

$$\begin{aligned}\frac{1}{\rho}\frac{\partial p}{\partial x} &= \frac{1}{\rho_2}\frac{p_b - p_a}{\Delta X}\\ &= \frac{1}{\rho_2 \Delta X}[(\rho_1 g Z_1 + \rho_2 g \Delta Z + \rho_2 g Z_2) - (\rho_1 g Z_1\\ &\quad + \rho_1 g \Delta Z + \rho_2 g Z_2)]\\ &= g\,\frac{\rho_2 - \rho_1}{\rho_2}\frac{\Delta Z}{\Delta X}\\ &= g\left(\frac{\rho_2 - \rho_1}{\rho_2}\right) i_2\end{aligned} \tag{6.5}$$

where i_2 is now the slope of the interface. There is no horizontal pressure gradient in the upper layer, and the pressure gradient calculated above applies throughout the lower layer.


i1
ΔZ
ρ
Z
ΔX
a
(a)
b
ρ1
Z1
i2
ΔZ
ρ2
Z2
ΔX
a
(b)
b

BALANCE OF FORCES: WITH CORIOLIS FORCE

In the following sections we examine in some detail the consequences of the Coriolis term to the equation of motion. With geostrophic motion, we balance the Coriolis force against the pressure-gradient force; with inertial motion, it is Coriolis force against particle acceleration; with Ekman motion, it is Coriolis force against wind stress. In the last two sections we looked at some simple examples of balancing the Coriolis force against more than one term. In each case the resulting flow is contrary to one's intuitive sense of what should happen. Even for those with considerable sophistication in physical concepts, one's first introduction to the consequences of the Coriolis force often produces something analogous to intellectual trauma.

Geostrophic Flow

Assume that the ocean currents are horizontal and steady, and that the wind stress and other frictional forces are sufficiently small that they may be neglected. With the acceleration and friction terms removed, Eq. (5.26) reduces to the hydrostatic equation in the vertical plane, and a balance between two terms, the pressure force and the Coriolis force, in the horizontal plane.

$$\begin{aligned} fv &= \frac{1}{\rho}\frac{\partial p}{\partial x} \\ fu &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \end{aligned} \tag{7.1}$$

This is the geostrophic equation, and currents that obey this relation are called *geostrophic currents* All the major currents in the ocean, such as the Gulf Stream, the Antarctic Circumpolar Current, and the equatorial currents are, to a first approximation, geostrophic currents. The consequences of the geostrophic equation are extraordinary. Consider for a moment again the problem of the rolling ball on a frictionless inclined plane. Substituting the slope force for the pressure gradient force,

$$fv = gi_x \tag{7.2}$$

results in a ball that rolls parallel to the slope (Figure 7.1). No matter what the direction of the slope, the ball rolls parallel to the slope. The geostrophic equation says water does not run downhill; it runs around the hill.

Television has made nearly everyone familiar with weather maps. The winds on weather maps do not blow directly from high-pressure cells to low-pressure cells. The winds' flow is more nearly parallel to the isobars. In the Northern Hemisphere the flow is clockwise around high-pressure cells and counterclockwise around low-pressure cells (Figure 7.1). A good rule of thumb: as you look downstream, the high pressure is on the right in the Northern Hemisphere and on the left in the Southern Hemisphere.

The forces involved in this balance are very small, generally less than $\frac{1}{100}$ dyne/g. Compare this with the hydrostatic force balance, which at 100 m is about 10^7 dynes/g and 5×10^8 dynes/g at 5000 m. But small as they are, the pressure-gradient and Coriolis forces are the largest horizontal forces in much of the ocean. All the major ocean circulation features are in approximate geostrophic balance.

The Gulf Stream flow and all other major surface currents are maintained by the slope of the sea surface. The sea-surface slope

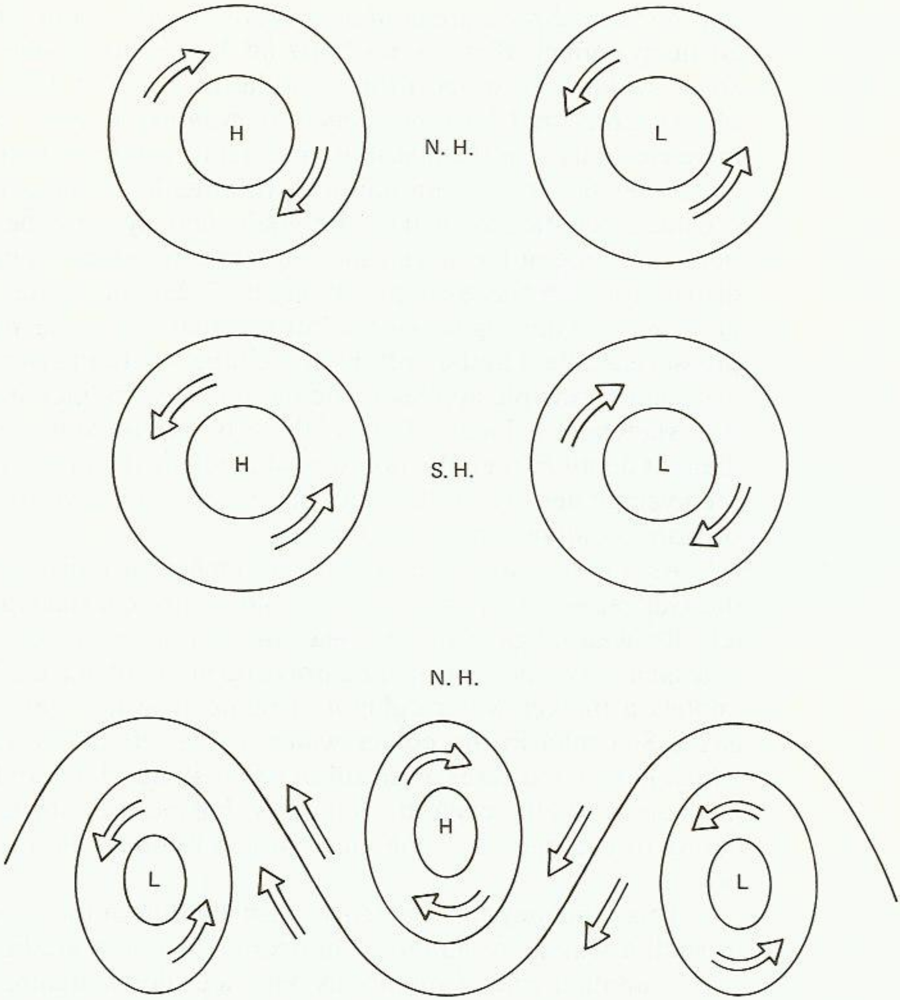


Figure 7.1. Geostrophic flow is *cyclonic* around low-pressure cells and *anticyclonic* around high-pressure cells. In the Northern Hemisphere, cyclonic flow is counterclockwise; it is clockwise in the Southern Hemisphere.

necessary to maintain the Gulf Stream is about 1 part in 100,000. Sea level at Bermuda is about 1 m higher than on the East Coast of the United States as a consequence.

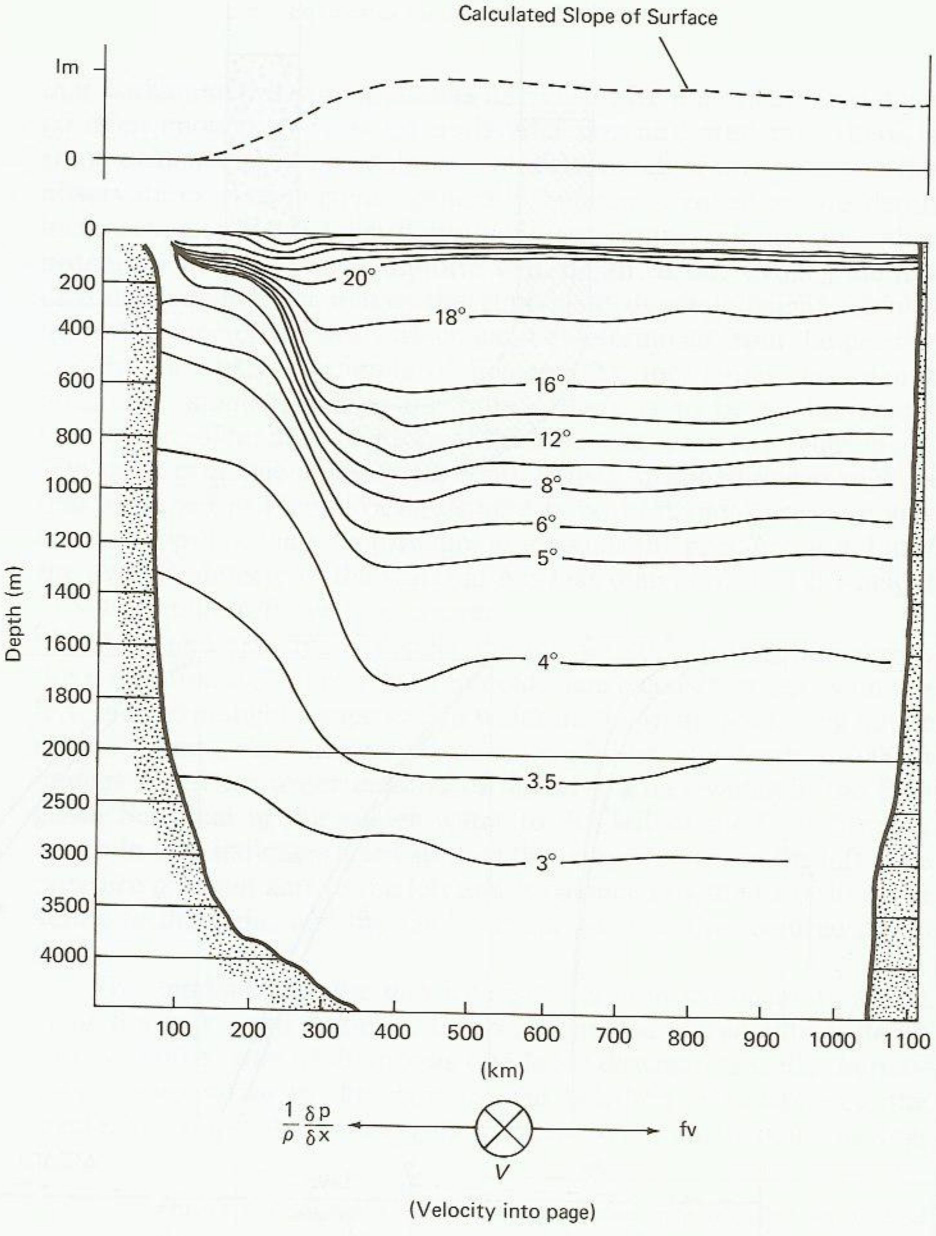


Figure 7.3. Typical temperature distribution across the Gulf Stream. If one assumes no horizontal pressure gradient at 4000 m, then the slope of the sea surface is as indicated. A geostrophic surface current to balance that pressure gradient would be into the paper as shown. (After Iselin, C. O'D., 1936: "A Study of the Circulation of the Western North Atlantic," *Papers in Physical Oceanography and Meteorology*, Vo. 4, No. 4.)

Wind Stress: Ekman Motion and Upwelling

When the wind blows along the surface of the ocean, it causes both surface currents and waves. The quantitative details of how the stress of the wind is applied to the ocean surface are not very well understood. Energy is transferred by some kind of turbulent process, and fuller understanding requires a detailed examination of not only the mean wind, current, and pressure fields, but of the variations of the wind, current, and pressure about the mean.

Several semiempirical observations are useful. One is that the surface current induced by a wind is approximately 3% of the wind. One might expect a 0.6-knot surface current with a 20-knot wind. The second is that the stress τ applied to the sea surface increases as the square of the wind speed:

$$\tau \cong 0.02W^2 \tag{7.7}$$

where W is the wind speed in meters per second and τ is the wind stress in dynes per square centimeter. A 10-m/s wind (nearly 20 knots) causes a wind stress of approximately 2 dynes/cm^2.

In fact, neither relationship is that simple. The "constant" varies with wind speed and surface roughness and depends critically upon how far above the sea surface the wind is measured. For winds measured at "deck height," 10–20 ft above the sea surface, Eq. (7.7) is good to within a factor of 2 and probably considerably better than that.

This kind of motion is called *Ekman motion,* after V.W. Ekman, who first examined the problem in 1902. Ekman's attention was called to it by Fridtjof Nansen, who, while frozen in the Arctic aboard the

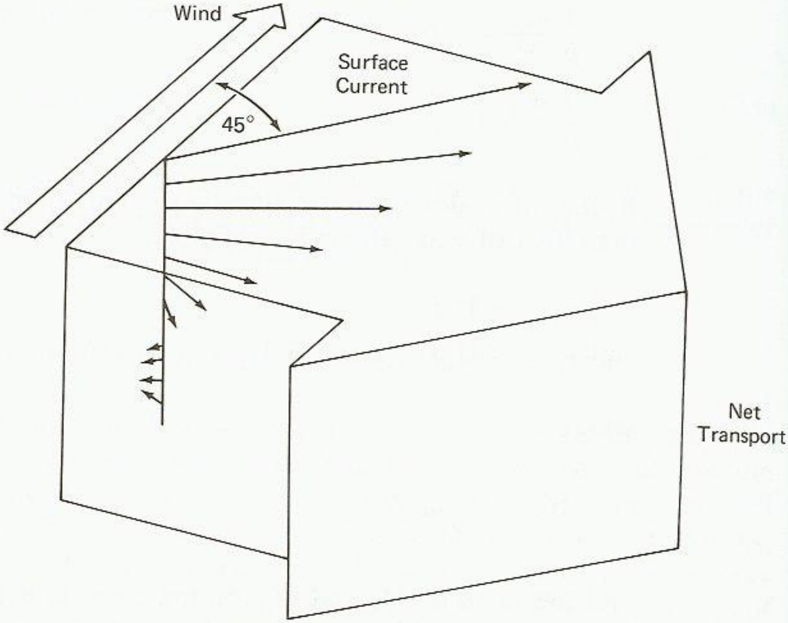


Figure 7.6. Water is set in motion by the wind. According to the Ekman relation, the effect of the Coriolis force is for each succeeding layer of water to move slightly to the right (in the Northern Hemisphere) of the one above it. The result is the Ekman spiral as shown with the net transport at 90° to the wind.

Fram, observed that when the wind blew the ice appeared to move not downwind, but at an angle of 20–40° to the right of the wind.

The variety of forces acting in the surface layer of the ocean is such that it seems unlikely that one would very often expect to observe a simple Ekman spiral, as in Figure 7.6. However, the observational evidence for a wind-driven surface current to move at some angle to the right of the wind is quite good, as is the evidence for a mass transport of water to the right of the wind, as in Eq. (7.9).

Upwelling is the term used in oceanography to describe the process by which deep water is brought to the surface. It has an importance well beond its physical significance, because the deep water carries nutrients to the surface layers where they can be assimilated by phytoplankton. Regions of upwelling are among the richest biological areas in the world.

In a formal sense, upwelling occurs wherever there is a divergent flow at the surface. Continuity requires upward vertical flow to replace the water lost by the surface divergence. The most famous

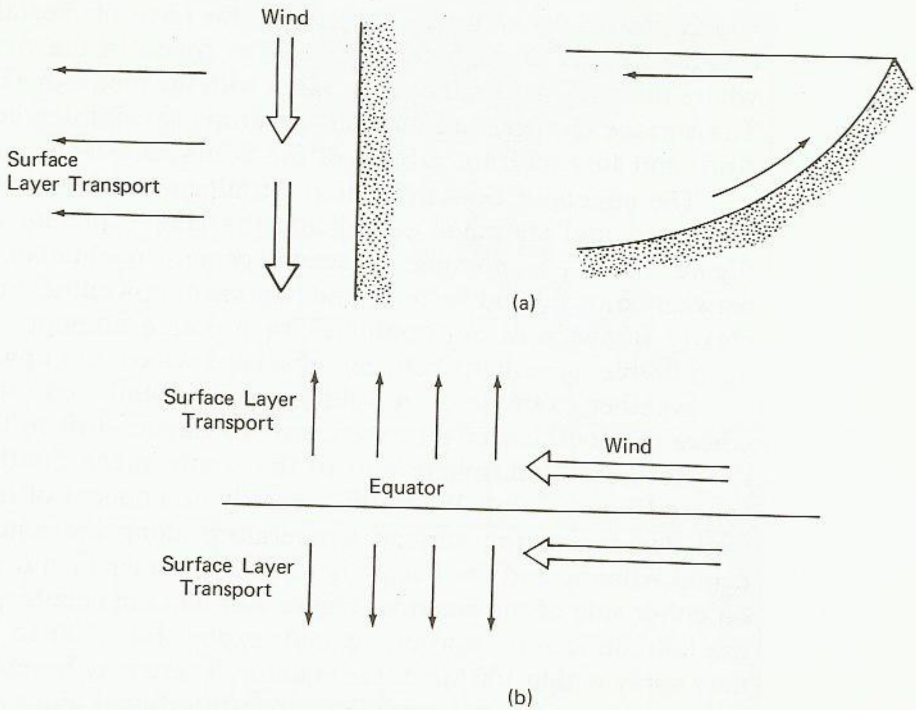


Figure 7.7. (a) Wind blowing parallel to the coast will transport the surface water offshore. This surface water will be replaced by colder water "upwelling" from below; (b) the effect of an easterly wind near the equator is to cause poleward transport of the surface water, which is replaced by colder water brought to the surface along the equator.

upwelling areas are those along certain coasts where the wind drives the surface water offshore. According to Ekman theory, the effect of wind is to drive the water to the right of the wind (in the Northern Hemisphere). Thus maximum upwelling occurs when the wind is parallel to the shore, not offshore (Figure 7.7a). Such a wind condition occurs, among other places, off the coasts of California and Peru. Colder than average surface water and higher than average biological productivity are the effects of upwelling. The region off Peru supports the highest-volume fishery in the world; this high productivity is related less to the cold rich surface waters brought from the Antarctic than it is to the wind-induced upwelling along the coast.

An examination of mean wind charts allows one to predict areas of coastal upwelling; these predictions can be verified by an examination of surface temperature charts. The cold coastal waters of Peru and California can only be explained on the basis of coastal upwelling (Figure 7.8). A striking example can be found in the Arabian Sea, where the seasonal wind pattern shifts with the monsoon (Figure 7.9). The surface temperature off Somalia drops several degrees between April and July with the advent of the Southwest Monsoon.

The effects of bottom friction, stratification, local currents, topography, and shoreline complicate the simple picture implied by Figure 7.7a. It is possible to secure general qualitative agreement between known wind systems and regions of upwelling, but the complexity of the problem continues to frustrate attempts at detailed quantitative agreement between observed winds and upwelling.

Another example of upwelling can be found along the equator, where the southeasterly trades cause an Ekman drift to the north in the Northern Hemisphere and to the south in the Southern Hemisphere (Figure 7.7b). Where the easterly component of the trades is well developed, the surface temperature along the equator in the cental Atlantic and the Pacific is often 2°C cooler than it is 100 miles on either side of the equator (Figure 7.8). On the equator, the sine of the latitude is zero, and one cannot expect Eq. (7.8) to hold, but it may apply within 100 km of the equator. There may be another explanation for the lowered surface temperature found along the equator (see the discussion of equatorial undercurrents in Chapter 8); but the evidence to date suggests that it is at least in part the result of a divergence of the surface waters resulting from a poleward Ekman transport on both sides of the equator.